A BRIEF DISCUSSION ON DARBOUX THEOREM AND ITS APPLICATIONS[1]

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Abstract

Every real-valued continuous function on a closed interval $[a, b]$ *attains all the values in between* $f(a)$ *and* $f(b)$ *known to be the Intermediate value theorem whereas, the Darboux theorem states that the derivative of a differentiable function on a closed interval has Intermediate value property, which need not to be continuous. This leaves us with uncertainty about the functions satisfying the Intermediate value property whether they possess a continuous mapping or not. Thus, a huge class of functions with Intermediate value property, explicitly, the derivative functions of differentiable functions was offered by the Darboux theorem and hence, the paper entails an extensive discussion of some of the proofs of the theorem arranged in an intuitive order. Most of them include the Extreme value property, along with various versions of the Mean value theorem, which provides a better understanding of the Darboux theorem and its applications through a reasonable discourse of concepts already established by mathematicians.*

Keywords: Darboux theorem; derivative; differentiable function; Intermediate value property; Mean value theorem; Rolle's theorem.

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Introduction

The Darboux theorem was established by Jean Gaston Darboux in 1875, and since then, it has proved its relevance, time and again. It forms the basis of various results in mathematics which makes it a crucial topic of analysis and research. Darboux challenged the earlier belief of mathematicians that a function that satisfies the Intermediate value property needs to be continuous, where a function f is said to have the Intermediate value property if it takes all values to lie in between $f(a)$ and $f(b)$ for any two points a and b in its domain. He also proved the existence of a discontinuous derivative of a differentiable function, which satisfies the Intermediate value property, along with various examples to justify his belief.

Darboux's proof is well-known in the course of Real Analysis. Many mathematicians follow their intuition and have tried to prove the theorem through various approaches. In this paper, we will incorporate different proofs, approaches, and applications of the Darboux theorem. The discussion is divided into three sections - the first section covers preliminaries that form the basis of the Darboux theorem, then the next section includes proofs by Lars Oslen, M. Bhandari, and Sam B. Nadler, which follow different pathways; finally, the paper is concluded by mentioning some of its applications. This paper can be used as a tool to gauge further studies in the domain and can be helpful to understand the theorem more clearly.

Preliminaries

Simply, a function with a continuous graph, without any breaks or jumps, is termed as a continuous function. There are some precise mathematical statements concerning the properties of continuous functions. It all started with the Extreme value theorem (Kumar & Kumaresan, 2014), which states that if a real-valued continuous function is defined on a closed interval then it attains both a maximum and minimum value in its domain. This theorem leads us to another important theorem of Calculus, that is, Rolle's Theorem (Bartle & Sherbert, 1992), which states that if a function f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) such that $f(a) = f\ (b)$, then $f'(x) = 0$ for some $x \in (a,b)$. The Rolle's theorem is a special case of the Mean value theorem (Bartle & Sherbert, 1992), where the Mean value theorem states that if f is a continuous function on the closed interval $[a, b]$, which is differentiable on the open interval (a, b), then there is a point $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.

The fact that Rolle's theorem was presented first in the 1690s and the general case took so long to come out became the bone of contention among various mathematicians. Even John Bernoulli questioned Rolle's understanding of Calculus in the *argumentum ad hominem*. However, both Rolle's and Mean value theorems play an important role in establishing many crucial concepts of this article.

Various Proofs of Darboux Theorem

The well-known Intermediate value theorem proved in 1817 states that if a function is continuous over a closed interval $[a, b]$, then it takes every value in between $f(a)$ and $f(b)$. Whereas, while dealing with its converse, a natural question arises: Is there any discontinuous function that satisfies the Intermediate value property? The French mathematician Jean Gaston Darboux provided the proof of the following result and gave an example of a differentiable function defined on an open interval with a discontinuous derivative satisfying the Intermediate value property. The result is referred to as the Darboux theorem (Darboux, 1878). We now present various proofs of this theorem.

Darboux Theorem: *Let* f *be a real-valued differentiable function defined on* (a, b) *and let* $c \leq d$ *are the points that lie in between* a *and* b. *If* y *is a point in between* $f'(c)$ *and* $f'(d)$ *then there exists a number* $x \in [c, d]$ *such that* $f'(x) = y$.

Proof I: For points $c \leq d$ in (a, b), define a function $\phi : [c, d] \to R$ by

$$\phi(t) = f(t) - yt.$$

The function ϕ is differentiable on $[c, d]$ as f is a differentiable function by hypothesis. Thus, $\phi'(t) = f'(t) - y$. Hence, $\phi'(c) = f'(c) - y$ and $\phi'(d) = f'(d) - y$, which implies that $\phi'(c)\phi'(d) < 0$. Without loss of generality, assume that $\phi'(c) < 0$ and $\phi'(d) > 0$. This implies that neither on c nor on d the function ϕ attains its local maximum value. Since ϕ is continuous and defined on a closed interval, it always attains its maximum on $[c, d]$ by the Extreme value theorem. Thus, there is a point $x \in (c, d)$ such that $\phi'(x) = 0$, that is, $f'(x) = y$.

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The argument that $\phi'(c) < 0$ and $\phi'(d) > 0$, involved in the above proof, implies that neither on c and nor on d the function ϕ attains its maximum value, which is proved by the $\varepsilon-\delta$ approach that adds to the discontent of the readers. Thus, a new proof of the Darboux theorem has been provided by Lars Olsen using the Mean value theorem and the Intermediate value theorem, (Oslen, 2004).

Proof II: Let y be a point strictly lying in between $f'(c)$ and $f'(d)$. Then define the continuous real valued functions f_c and f_d on interval (a, b) as

$$f_c(t) = \{f'(c), \quad t = c \ \frac{f(c)-f(t)}{c-t}, \quad t \neq c$$

and

$$f_d(t) = \{f'(d), \quad t = d \ \frac{f(t)-f(d)}{t-d}, \quad t \neq d$$

which implies that $f_c(c) = f'(c)$ and $f_c(d) = \frac{f(c)-f(d)}{c-d}$, with $f_d(d) = f'(d)$ and $f_d(c) = \frac{f(c)-f(d)}{c-d}$.

Thus, either $y \in (f_c(c), f_c(d)]$ or $y \in [f_d(c), f_d(d))$. When y lies in $(f_c(c), f_c(d)]$, the continuity of f_c implies the existence of a number x in $[c, d]$ such that

$$y = f_c(x) = \frac{f(x)-f(c)}{x-c}.$$

By Mean value theorem, there exists a number e in $[c, x]$ such that

$$y = \frac{f(x)-f(c)}{x-c} = f'(e).$$

Similarly, when y lies between $f_d(c)$ and $f_d(d)$, the same argument ensure the existence of a point e in $[c, d]$ such that

$$y = \frac{f(d)-f(c)}{d-c} = f'(e)\,. \qquad \blacksquare$$

This proof is more convincing as it avoids the use of ε–δ arguments. Lars Olsen has used the Caratheodory's theorem for the existence of the auxiliary functions f_c and f_d. But the use of this theorem for the existence of the functions f_c and f_d is perhaps too strenuous for a student beginning the study of real analysis at the undergraduate level.

The following proof, introduced by Mukta Bhandari (Bhandari, 2016), uses the monotonicity property of a differentiable function, the Intermediate value theorem, and the Rolle's theorem while avoiding ε–δ arguments. The use of standard and familiar concepts makes this proof simpler to understand at an elementary level.

Proof III: Again, considering the function $\phi : [c, d] \to R$ defined by $\phi(t) = f(t) - yt$ where f is differentiable and assuming that $f'(c) < y < f'(d)$. Then

$$\phi'(t) = f'(t) - y.$$

Hence, $\phi'(c) = f'(c) - y < 0$ and $\phi'(d) = f'(d) - y > 0$.

Since the function ϕ is increasing as well as decreasing at different intervals of its domain, the function is non-monotonic on $[c, d]$. Therefore, there exists $\alpha, \beta, \gamma \in [c, d]$ such that $\alpha < \beta < \gamma$ such that exactly one of the following holds:

(a) $\phi(\alpha) < \phi(\beta)$ and $\phi(\beta) > \phi(\gamma)$

(b) $\phi(\alpha) > \phi(\beta)$ and $\phi(\beta) < \phi(\gamma)$

For part (a), there are following three cases:

(1) $\phi(\alpha) < \phi(\gamma)$

(2) $\phi(\alpha) > \phi(\gamma)$

(3) $\phi(\alpha) = \phi(\gamma)$

Case 1: Assume that $\phi(\alpha) < \phi(\gamma)$. Then $\phi(\alpha) < \phi(\gamma) < \phi(\beta)$. As f is differentiable by hypothesis, the differentiability of ϕ implies its continuity on $[c, d]$. Therefore, $\phi([\alpha, \beta]) \subset [c, d]$ by Intermediate value theorem. Thus, there exists $x \in (\alpha, \beta) \subset (c, d)$ such that $\phi(x) = \phi(\gamma)$ where $c \leq \alpha < x < \beta < \gamma \leq d$.

Now, by applying Rolle's Theorem on ϕ for the closed interval $[x, \gamma]$, there is a number $u \in (x, \gamma)$ such that $\phi'(u) = 0$ where $(x, \gamma) \subset (\alpha, \gamma) \subset (c, d)$, implies that $f'(u) = y$.

Case 2: When $\phi(\alpha) > \phi(\gamma)$, we have $\phi(\gamma) < \phi(\alpha) < \phi(\beta)$. Then the result holds as in case 1.

Case 3: When $\phi(\alpha) = \phi(\gamma)$, then by Rolle's theorem again, we have a point $x \in (\alpha, \gamma) \subset (c, d)$ such that $\phi'(x) = 0$. Thus, we get $f'(x) = y$.

We obtain the same result in case (b) by following similar arguments.

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This proof manages to encompass all elements that seemingly enabled insufficiencies in the proofs presented earlier. However, we will continue this discussion by providing yet another proof of Darboux's theorem by Sam B. Nadler, Jr., (Nadler, 2010). This doesn't intend to improve on the preceding proofs but rather propounds itself as a noteworthy addition to our discourse.

Proof IV: Let $f : (a, b) \to R$ be a differentiable function with A be the set of all values of the derivative of f and B be the set of all slopes of chords joining points on the graph of f. It is claimed that A is an interval, that is, f' satisfies the Intermediate value property.

Clearly, the set $B \subset A$ by the Mean value theorem, and the definition of derivative implies that $A \subset \overline{B}$. Now, we are going to prove that B is an interval, which implies our claim, for which any two points of B can be connected by an interval in B. Let $x, y \in B$ be any two points. Then

$$x = \frac{f(p)-f(q)}{p-q} \text{ and } y = \frac{f(r)-f(s)}{r-s} \text{ where } p < q \text{ and } r < s \text{ in } (a,b).$$

Now, define a continuous function $h : [0,1] \rightarrow B$ by

$$h(t) = \frac{f((1-t)p+tr)-f((1-t)q+ts)}{((1-t)p+tr)-((1-t)q+ts)} .$$

Here, h gives the slopes of the chords with endpoints sliding over the graph of f such that $h(0) = x$ and $h(1) = y$. Therefore, $h([0,1])$ is an interval in B by the Intermediate value theorem. Hence, B is an interval.

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Until now, every proof that we have considered has employed a unique approach to the Darboux theorem; whereas, Nadler's proof employs a transparent and conceptual approach that systematically helps to visualise why the theorem is true. Thus, we conclude our discourse with this proof.

Applications of Darboux Theorem

1.1. Inter-relation between the Cauchy mean value theorem and the Mean value theorem.

The Cauchy Mean value theorem is the generalisation of the Mean value theorem. It states that if ϕ and ψ are continuous real-valued functions on $[a,b]$, which are differentiable on (a,b) with $\psi'(x) \neq 0$ for any $x \in (a,b)$, then there exists a number $c \in (a,b)$ such that

$$\frac{\phi(b)-\phi(a)}{\psi(b)-\psi(a)} = \frac{\phi'(c)}{\psi'(c)}$$

If the map t to $(\psi(t), \phi(t))$ from $[a,b]$ to R^2 is considered as a parameterised curve on the plane, then the Cauchy mean value theorem ensures the existence of a point

$(\psi(c), \phi(c))$ on the curve for some $c \in (a, b)$ such that the slope $\frac{\phi'(c)}{\psi'(c)}$ of the line tangent to the curve at that point is equal to the slope of the chord joining the endpoints of the curve.

When $\psi(x) = x$ in the Cauchy mean value theorem, it reduces to the Mean value Theorem.

The equivalent relation between the Cauchy mean value theorem and the Mean value theorem has been proved by André Pierro de Camargo (Camargo, 2020), using the Darboux theorem, as follows:

Since $\psi'(x) \neq 0$ for all $x \in (a, b)$, either $\psi'(x) > 0$ or $\psi'(x) < 0$. Thus the function ψ is invertible. Now, by applying the Mean value theorem on the function $\xi(x) = \phi o \psi^{-1}(x)$ with points $x_1 = \psi(a)$ and $x_2 = \psi(b)$, we get (Kumar & Kumaresan, 2014)

$$\frac{\phi(b)-\phi(a)}{\psi(b)-\psi(a)} = \frac{\xi(x_2)-\xi(x_1)}{x_2-x_1} = \xi'(\eta) = \phi'\left(\psi^{-1}(\eta)\right)\left[\psi^{-1}\right]'(\eta) = \frac{\phi'\left(\psi^{-1}(\eta)\right)}{\psi'\left(\psi^{-1}(\eta)\right)}$$

for $\psi^{-1}(\eta) \in (a, b)$.

1.2. The derivative of a piecewise function cannot have jump discontinuities.

In calculus, we come across several exercises that involve calculating the derivative of piecewise-defined functions, which are defined by multiple sub-functions where each sub-function is applicable to a different interval in the domain. The aim behind such exercises is to consolidate new learners and their ability to apply the definition of the derivative. Generally, these kinds of functions are made up of two smooth pieces joined together at a single point. The strategy that avoids using the definition of derivative is to find the derivative function of each smooth piece and then check whether they agree at the selected point or not. It is beyond the scope of a calculus course to show that the strategy mentioned above works together with investigating discontinuities of the derivative.

Under this application of the Darboux theorem introduced by B Tomas Johansson (Johansson, 2016), we primarily discuss the method used to find the derivative of piecewise functions and focus on the proof that there seems to be no function that has a derivative with jump discontinuity.

The discussion begins with the following example from (Johansson, 2016), where we look into the method that is used to find the unknown value, k, in this case. Consider the real-valued function ϕ defined by

$$\phi(x) = \{1 + kx + x^2, \quad x < 0 \ \cos cos\,(x), \quad x \geq 0$$

whose derivative exists at $x = 0$.

To find the value of k, we first find the derivative function of each piece in the above equation and determine the value of k, for which these functions agree at $x = 0$, instead of setting up the definition of the derivative.

In the above case, this strategy works as each piece in the function is smooth and also defined at the origin. But, it will not always work in general. Now, coming towards the next example where we will check whether the function ψ defined by

$$\psi(x) = \{x^2 \sin sin\left(\frac{1}{x}\right), \quad x < 0 \ x^2, \qquad x \geq 0$$

has a derivative at $x = 0$ or not. This example is often presented to steer away from the above strategy where the definition of the derivative is applied. As the derivative of $\psi(x)$ when $x < 0$, comes out to be

$$\psi^{'}(x) = 2x\, sin\left(\frac{1}{x}\right) - cos\left(\frac{1}{x}\right),$$

the limit in the above case does not exist. But, if we go by the definition of the derivative, then it can be concluded that the function is indeed differentiable at the origin and has a derivative equals to zero.

The problem addressed above raised a very crucial question regarding the existence of the derivative at the given point where the left-hand side derivative function approaches the same limit as the right-hand side derivative function whenever x approaches 0 or, to some other point where the pieces are joined. Now, to answer this, Johansson gave some elementary arguments using the Mean value theorem and proved that if ϕ is a continuous function on (m, n) and differentiable on whole (m, n) except at a point x_0 where $\phi'(x) \; = \phi'(x) \; = L$, then ϕ is differentiable at x_0 as well, with $\phi'\left(x_0\right)$ = L.

Now, we will discuss the proof that uses the Darboux theorem to show that derivative functions do not contain jump discontinuities. Let ϕ be a differentiable function on the interval $[m, n]$ and M be a midpoint of the given interval. Assuming that $\alpha = 2x - m$ and $\beta = 2x - n$ are two smooth monotonically increasing functions where α is defined from $[m, M]$ to $[m, n]$ and β is defined from $[M, n]$ to $[m, n]$. Then, by (Johansson, 2016), the whole function is defined as follows:

$$\xi(x) = \{\phi'(m), \qquad x = m \, \frac{\phi(\alpha(x)) - \phi(m)}{\alpha(x) - m}, \quad m < x < M \, \frac{\phi(n) - \phi(\beta(x))}{n - \beta(x)}, \quad M < x < n \, \phi'(n), \qquad x = n$$

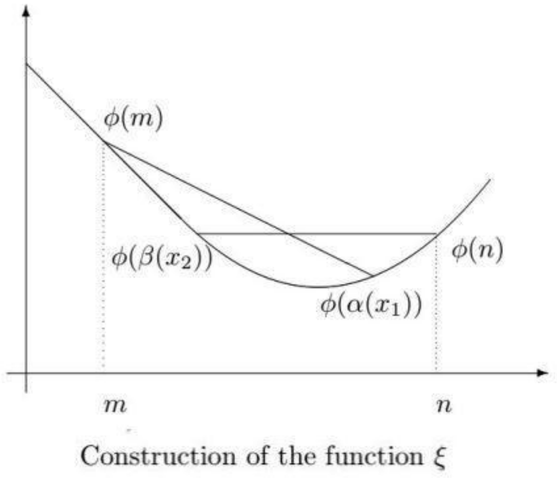

Figure 1

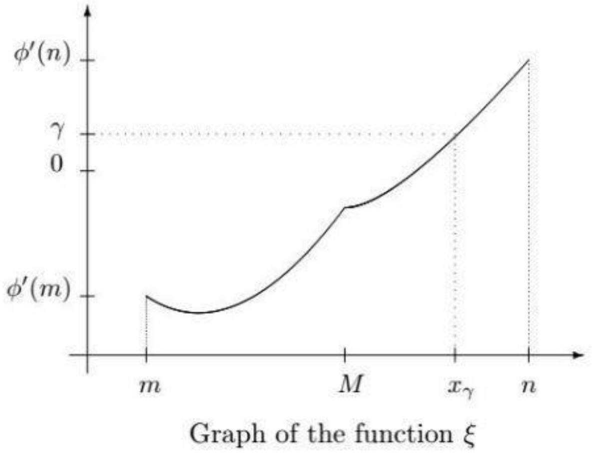

Figure 2

Since, $\alpha(M)$ and $\beta(M)$ are equal to n and m respectively, then ξ can be referred as a continuous function at $x = M$. As $x \to m^{+}$ implies $\alpha(x) \to m$ and by the definition of derivative, the function is said to be continuous at $x = m$. Similarly, the continuity of the function at $x = n$ can be proved. Here, $\xi(x)$ is the slope of a line segment having endpoints on the curve that depends on the point in the interval (m, n). The figures given above can be referred to get a better understanding of the function ξ (see Figure 1) and its graph (see Figure 2).

By taking γ to be a real number that lies between $\phi^{'}(m)$ and $\phi^{'}(n)$, continuity of ξ implies the existence of a point x_{γ} where $\xi\left(x_{\gamma}\right) = \gamma$. Without the loss of generality, $M < x_{\gamma} < n$, is assumed with $\beta(x_{\gamma}) = x$ and by the definition of the value of $\xi(x_{\gamma})$ from the above defined function, we get

$$\frac{\phi(n)-\phi(x)}{n-x} = \gamma.$$

As per the Mean value theorem, there is a point say, $\eta \in (x, n)$ where

$$\frac{\phi(n)-\phi(x)}{n-x} = \phi^{'}(\eta).$$

From the above two equations, it can be concluded that there exists a point η where the value of $\phi^{'}(\eta)$ lies between the values of $\phi^{'}(m)$ and $\phi^{'}(n)$.

Hence, it is proved that there exists no function whose derivative has a jump discontinuity. This proof clears many doubts that a new learner might encounter. It also gives the readers a new and broader explanation of the concept. It thereby makes this application of the Darboux theorem significant while dealing with the derivative of piecewise functions.

Conclusion

The study of this paper enables us to understand the Darboux Theorem in a better way through various approaches along with different versions of the Mean value theorem. It also paves the way to multi-directional aspects of thinking to establish theorems along with their proofs in real analysis which is a base to many fundamental concepts of the domain and can act as an important tool in exploring many advanced results of the subject.

One concept of mathematics opens a door for another and hence, this study of Darboux theorem in real analysis offers a base that will help us to unfold many crucial results of topology.

Talking about some of the existing studies, the first that comes in mind is the one standard example:

$$f(x) = \ \{\sin sin\left(\frac{1}{x}\right), \qquad x\neq 0 \ \ 0, \qquad\qquad x \ = \ 0$$

which falsifies the claim of reverse Intermediate value theorem. Since, the function is discontinuous only at $x = 0$, it is termed as a weak counter example. This problem was introduced by Greg Oman in his paper **"The Converse of the Intermediate value theorem: from Conway to Cantor to Cosets and beyond"** (Oman, 2014). The author presented several methods of construction of strong counter examples, then presented various ways to prove the existence of such functions along with some more detailed discussions. This can further lead us to the study of closed graph theorem (Marquina, 1977), which includes several conditions where a function can have a closed graph and has a continuous mapping. It also paves the way of discussion of various implications of the Intermediate value theorem and its converse wherein some assumptions along with additional suppositions are used to derive various results and showcase continuity, closed graph as well as the satisfying the Intermediate value theorem in some Hausdorff Topological space (Kabbouch & Najmeddine, 2021).

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